For two conts. r.v.'s X and Y,

Joint pdf: $f(x,y)$.

Requirements: ① $f(x,y) \geq 0 \quad \forall (x,y) \in \mathbb{R}^2$

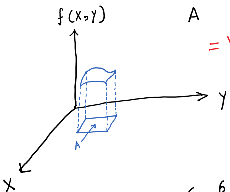
② $\iint_{\mathbb{R}^2} f(x,y)\,dy\,dx = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f(x,y)\,dy\,dx = 1.$

Joint Support: $D = \{(x,y) : f(x,y) > 0\}$.

$P((X,Y)\in A) = \iint_A f(x,y)\,dy\,dx$ [A is a set of points (x,y)]

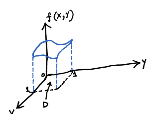
= Volume that the surface of $f(x,y)$ encompasses above A in the XY-plane

$\in [0,1]$ $\quad [P((X,Y)\in\mathbb{R}^2) = 1]$



Eg 2: $f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & \text{, if } 0\le x\le 1 \text{ and } 0\le y\le 1 \\ 0 & \text{, o.w.} \end{cases}$

$D = \{(x,y) : 0\le x\le 1,\ 0\le y\le 1\} = [0,1]\times[0,1] = [0,1]^2$: Unit Square in $\mathbb{R}^2$



① $f(x,y) \geq 0$

② $\iint_{\mathbb{R}^2} f(x,y)\,dy\,dx = \int_0^1\int_0^1 \frac{6}{5}(x+y^2)\,dy\,dx = 1$ (Check)

$P(0\le X\le \frac{1}{4}, 0\le Y\le\frac{1}{4}) = \int_0^{1/4}\int_0^{1/4}\frac{6}{5}(x+y^2)\,dy\,dx = \frac{7}{640}$ (check)

$P(X=\frac{1}{4}, Y=\frac{1}{4}) = 0$

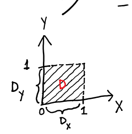
$P(X=\frac{1}{4}, 0\le Y\le\frac{1}{4}) = 0$

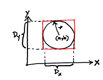
$P(\frac{1}{4}\le Y\le\frac{3}{4}) = P(-\infty < X < \infty, \frac{1}{4}\le Y\le\frac{3}{4}) = \int_{-\infty}^{\infty}\int_{1/4}^{3/4} f(x,y)\,dy\,dx = \int_0^1\int_{1/4}^{3/4}\frac{6}{5}(x+y^2)\,dy\,dx = \frac{37}{80}$ (check)

Marginal Supports: $D_X = [0,1] = D_Y$

Here, $D = D_X \times D_Y$

(Rectangular Support)





Example of Non-rectangular Support

$D_X\times D_Y = \{(x,y) : x\in D_X, y\in D_Y\} = \overbrace{[a-r, a+r]}^{D_x}\times\overbrace{[b-r, b+r]}^{D_y}$

$D = \{(x,y) : (x-a)^2+(y-b)^2 \le r^2\}$: Circle with center at (a,b) and radius r

(NOT rectangular)

Marginal pdf:

$f_X(x) \stackrel{\text{defn.}}{=} \int_{y\in D_Y} f(x,y)\,dy$

$f_Y(y) \stackrel{\text{defn.}}{=} \int_{x\in D_X} f(x,y)\,dx$

Eg 2. $f_X(x) = \begin{cases} \int_0^1 \frac{6}{5}(x+y^2)\,dy = \frac{6x}{5} + \frac{6}{5}\cdot\left[\frac{y^3}{3}\right]_0^1 = \frac{6x+2}{5} & \text{, if } x\in[0,1] \\ 0 & \text{, if } x\notin[0,1] \end{cases}$

$f_X(x)\geq 0, \quad \int_0^1 f_X(x)\,dx = \int_0^1 \frac{6x+2}{5}\,dx = 1$ (check)

Similarly, $f_Y(y) = \begin{cases} \frac{6y^2+3}{5} & \text{, if } y\in[0,1] \\ 0 & \text{, o.w.} \end{cases}$ (check)

Check that it's a density.

$P(\frac{1}{4}\le Y\le\frac{3}{4}) = \int_{1/4}^{3/4} f_Y(y)\,dy = \int_{1/4}^{3/4}\frac{6y^2+3}{5}\,dy = \frac{37}{80}$ (check)

$\left[\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{x=-\infty}^{\infty}\left\{\int_{y=-\infty}^{\infty} f(x,y)\,dy\right\}dx = 1\right]$

• Independence:

X, Y discrete: $A_x = \{X=x\}$, $B_y = \{Y=y\}$

X and Y are indep. if A_x and B_y are indep. $\forall (x,y)$.

Defn: X and Y are indep. if A_x and B_y ... $P(B_y)$

equivalently, $P(A_x \cap B_y) = P(A_x) \cdot P(B_y)$

$\iff p(x,y) = p_X(x) \cdot p_Y(y), \ \forall (x,y)$

X,Y conts: Indep. iff $f(x,y) = f_X(x) \cdot f_Y(y), \ \forall (x,y)$

If X and Y are indep., $P(a \le X \le b, c \le Y \le d) = P(a \le X \le b) \cdot P(c \le Y \le d)$

Pf: $LHS = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \int_c^d f_X(x) \cdot f_Y(y)\,dy\,dx$

$= \int_a^b f_X(x) \cdot \left[\int_c^d f_Y(y)\,dy\right] dx$

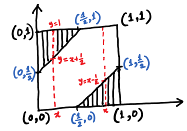
$= \left[\int_a^b f_X(x)\,dx\right]\left[\int_c^d f_Y(y)\,dy\right] = RHS$

Extend for discrete case!

In general, $\underbrace{P(X \in A, Y \in B)}_{P((X,Y) \in A \times B)} = P(X \in A) \cdot P(Y \in B)$ if X and Y are indep.

Eg2: Same Joint density as yesterday.

$$P\left(|X-Y|>\tfrac{1}{2}\right) = P\left(X>Y+\tfrac{1}{2}\right)+P\left(Y>X+\tfrac{1}{2}\right)$$



$y=x+\frac{1}{2}$ → Passes through $(0,\frac{1}{2})$ and $(\frac{1}{2},1)$

$x=y+\frac{1}{2}$ → Passes through $(\frac{1}{2},0)$ and $(1,\frac{1}{2})$

$$= \int_{x=0}^{1/2}\int_{y=x+\frac{1}{2}}^{1} f(x,y)\,dy\,dx + \int_{x=1/2}^{1}\int_{y=0}^{x-1/2} f(x,y)\,dy\,dx \quad (\text{finish})$$

- **Independence:**

Outside this rectangle both sides are 0.

$p(x,y) = p_X(x)\cdot p_Y(y)$, $\forall (x,y)$ (equivalently $\forall (x,y)\in \underbrace{D_X\times D_Y}$)

OR, $f(x,y) = f_X(x)\cdot f_Y(y)$, $\forall (x,y)$ (equivalently $\forall (x,y)\in D_X\times D_Y$)

→ Entry in the table = (Corresponding Row Total) × (Corresponding Col. Total)

Let's look at Eg 1, $p(100,0) = 0.2 \neq p_X(100)\,p_Y(0)$

$p_X(100) = 0.5$

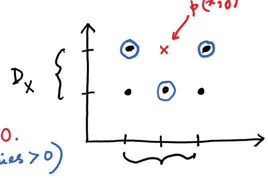
$p_Y(0) = 0.25$

$\Rightarrow$ X and Y are dep.

$\Rightarrow p(x,y) \neq p_X(x)\cdot p_Y(y)$ for any one (x,y).

- To show dep, show $p(x,y) \neq p_X(x)\cdot p_Y(y)$ for all (x,y).
- To show indep, show $p(x,y) = p_X(x)\cdot p_Y(y)$ for all (x,y).

Eg.



If we change eg1 so that entry at x is 0. (all other entries > 0)

At x, $p(x,y) = 0 \neq \underset{\overset{\vee}{0}}{p_X(x)}\cdot\underset{\overset{\vee}{0}}{p_Y(y)}$ ← These are true because of the ⊙ points.

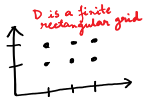
$\Rightarrow$ NOT indep.

Indep. $\Rightarrow$ Rectangular Joint Support
Non-rectangular Joint Support $\Rightarrow$ Dep.
} works for conts. case also.

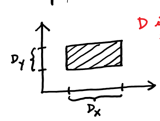
If you have rectangular Joint Support, Check pointwise for indep.

Rectangular Support:
(Examples of different cases)

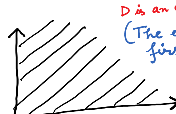
D is a finite rectangular grid



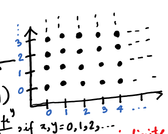
D is a bounded rectangle



D is an unbounded rectangle (The entire first quadrant)



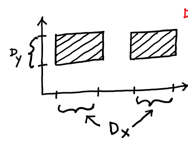
eg. $X\sim Poi(\lambda)$
$Y\sim Poi(\mu)$
X & Y are indep.



$$p(x,y) = p_X(x)\,p_Y(y) = \begin{cases} \dfrac{e^{-\lambda}\cdot\lambda^x}{x!}\cdot\dfrac{e^{-\mu}\cdot\mu^y}{y!}, & \text{if } x,y=0,1,2,\dots \\ 0, & \text{o.w.}\end{cases}$$

D is an infinite rectangular grid

D is a collection of bounded rectangles arranged in a grid shape



eg. $X\sim Exp(\lambda)$
$Y\sim Exp(\mu)$
X and Y are indep.

Then $f(x,y) = f_X(x)\cdot f_Y(y) = \begin{cases}\lambda e^{-\lambda x}\cdot \mu e^{-\mu y}, & \text{if } x,y>0\\ 0, & \text{o.w.}\end{cases}$

If I said: $f(x,y) \propto e^{-(\lambda x+\mu y)}$, $\forall (x,y)$ s.t. $x>0, y>0$,
and then ask you to check for indep.

Ans: Joint Support is first Quadrant → Rectangular (May or may not be indep.)

$\rightarrow$ Indep. (Need not find out $f_X(x)$ and $f_Y(y)$)

Next $f(x,y) \propto \underbrace{e^{-\lambda x}}_{f_1(x)}\cdot\underbrace{e^{-\mu y}}_{f_2(y)}$

Such that $\int_{-\infty}^{\infty} f_1(x) < \infty$ and $\int_{-\infty}^{\infty} f_2(x) < \infty$ } Just finding these two functions is enough for indep.

$f_X(x) = \int f(x,y)\,dy \propto \int f_1(x)\,f_2(y)\,dy = f_1(x)\cdot\int f_2(y)\,dy = K\,f_1(x)$ (where $K=\int f_2(y)\,dy$)

$\Rightarrow f_X(x) \propto f_1(x) \Rightarrow f_X(x) = K_1 f_1(x)$ (for some K_1)

Similarly, $f_Y(y) = K_2\,f_2(y)$ (for some K_2)

Now, $f(x,y) \propto f_1(x)\,f_2(y) \Rightarrow f(x,y) \propto f_X(x)\,f_Y(y)$ [$f_1(x)f_2(y)$ and $f_X(x)f_Y(y)$ differ by the constant $K_1\cdot K_2$ only]

$\iint f(x,y)\,dy\,dx = 1$ (as f is a joint density) ... $(\int f_X(x)\,dx) = 1\times 1 = 1$ (as both $f_X(x)$ and $f_Y(y)$ are densities)

Now, $f(x,y) \propto f_1(x)\, f_2(y) \Rightarrow f(x,y) \propto f_X(x)\, f_Y(y)$ [... by the constant $k_1 \cdot k_2$ only]

But since $\iint f(x,y)\, dy\, dx = 1$ (as f is a joint density)

and, $\iint f_X(x) f_Y(y)\, dy\, dx = \left(\int f_X(x)\,dx\right)\left(\int f_Y(y)\,dy\right) = 1 \times 1 = 1$ (as both $f_X(x)$ and $f_Y(y)$ are densities)

$f(x,y) = f_X(x) \cdot f_Y(y) = (k_1 f_1(x))(k_2 f_2(y))$.

So if you can write $f(x,y)$ as a product of two functions $f_1(x)$ and $f_2(y)$, you have essentially proved independence (provided the support is rectangular) and you have also found out the marginal densities $f_X(x)$ and $f_Y(y)$ without the proportionality constants.

In Eg2., Joint Support $= [0,1] \times [0,1]$: Rectangular

Rectangular Support does NOT guarantee indep.

But $f(x,y) = \frac{6}{5}(x+y^2)$ on $[0,1]\times[0,1]$

$$\neq \underbrace{f_X(x)}_{\frac{6x+2}{5}} \cdot \underbrace{f_Y(y)}_{\frac{6y^2+3}{5}}$$

Check at $f(1,1) = \frac{6}{5}(1+1^2) = \frac{12}{5}$

$f_X(1) = \frac{8}{5}$, $f_Y(1) = \frac{9}{5}$

$\Rightarrow f(1,1) \neq f_X(1) \cdot f_Y(1) \Rightarrow$ NOT indep.

- **Cond^{l} PMF:**

$$P(A_x \mid B_y) = P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{p(x,y)}{p_Y(y)} \quad [p_Y(y) > 0 \Leftrightarrow y \in D_Y]$$

Joint pmf

Marginal pmf

Cond^{l} PMF of Y given $\{X=x\}$ $\overset{\text{Not}^{n}}{:\equiv}$ $p_{Y|X}(y|x)$ $\rightarrow$ $P(Y=y \mid X=x) = \frac{p(x,y)}{p_X(x)} \quad [p_X(x) > 0 \Leftrightarrow x \in D_X]$

- **Cond^{l} PDF:**

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} \quad [x \in D_X]$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} \quad [y \in D_Y]$$

Lecture 27

Friday, July 21, 2017 1:00 PM

$p_{Y|X}(y|x)$ and $f_{Y|X}(y|x)$ give valid dist^{n}s. when you consider all y-values $\in D_Y$. [For each $x \in D_X$]

Similar results for $p_{X|Y}(x|y)$ and $f_{X|Y}(x|y)$.

- $p_{Y|X}(y|x) \geq 0 \ \forall x \in D_X$.
- $\sum_{y \in D_Y} p_{Y|X}(y|x) = \sum_{y \in D_Y} \frac{p(x,y)}{p_X(x)} = \frac{1}{p_X(x)} \cdot \sum_{y \in D_Y} p(x,y) = \frac{p_X(x)}{p_X(x)} = 1 \quad (\forall x \in D_X)$

Similar proof in conts. case and for cond^{l} dist^{n} of X given Y.

In Eg 1, $p_{X|Y}(100|0) = P(X=100|Y=0) = \frac{p(100,0)}{p_Y(0)} = \frac{0.2}{0.25} = \frac{4}{5}$

$p_{X|Y}(250|0) = \frac{p(250,0)}{p_Y(0)} = \frac{0.05}{0.25} = \frac{1}{5}$

$p_{X|Y}(x|0)$ is a valid pmf for X on $D_X = \{100, 250\}$

$p_X(100) = 0.5 = p_X(250)$

NOT the same dist^{n} as the cond^{l} dist^{n} of X given $Y=0$.

$\Rightarrow$ Dependent

Indep. $\Leftrightarrow$ $p(x,y) = p_X(x) \cdot p_Y(y) \quad \forall (x,y)$

$\Rightarrow p_X(x) = \frac{p(x,y)}{p_Y(y)} = p_{X|Y}(x|y)$ when $y \in D_Y$

and, $p_Y(y) = p_{Y|X}(y|x)$ when $x \in D_X$

Indep. of two r.v.'s is equivalent to saying that the cond^{l} dist^{n} of one variable given the other variable is fixed at some value is same as the marginal dist^{n} of the first variable.

- Cond^{l} CDF: $F_{Y|X}(y|x) = \int_{-\infty}^{y} f_{Y|X}(z|x)\, dz$
- Cond^{l} Mean: $E[Y|X=x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\, dy$

  $E[h(Y)|X=x] = \int_{-\infty}^{\infty} h(y) \cdot f_{Y|X}(y|x)\, dy$
- Cond^{l} Percentile: $\eta_{Y|X}(p|x)$: Soln. of $F_{Y|X}(y|x) = p$ (Solve for y)

  Hence $F_{Y|X}(\eta_{Y|X}(p|x)|x) = p$.
- Cond^{l} Var: $V(Y|X=x) = E[(Y - E(Y|X=x))^2 | X=x]$

  $= E[Y^2|X=x] - E^2[Y|X=x]$ (Check)

Compare all these cond^{l} quantities with corresponding uncond^{l} quantities, i.e. the marginal quantities.

eg. $F_Y(y), E(Y), E[h(Y)], \eta_Y(p), V(Y)$ etc.

Eg2: Given $X = 0.8$,

$f_{Y|X}(y|0.8) = \frac{f(0.8,y)}{f_X(0.8)} = \frac{\frac{6}{5}(0.8+y^2)}{\frac{6}{5} \times 0.8 + \frac{2}{5}}$ $\left[f_X(x) = \frac{6x+2}{5} I_{[0,1]}(x) \right]$

$= \begin{cases} \frac{30y^2+24}{34}, & \text{if } y \in [0,1] \\ 0, & \text{o.w.} \end{cases}$

(This is a valid density on $[0,1] = D_Y$ — Check)

Compare it with $f_Y(y) = \frac{6y^2+3}{5} I_{[0,1]}(y)$ ($\Rightarrow X, Y$ are dep.)

Compare it with $f_Y(y) = \frac{6y^2+3}{5} I_{[0,1]}(y)$

$$P(Y \le 0.5 \mid X = 0.8) = \int_0^{0.5} \frac{30y^2+24}{34}\, dy = 0.39 \text{ (Check)}$$

$$E(Y \mid X = 0.8) = \int_0^1 y \cdot \frac{30y^2+24}{34}\, dy = 0.574 \text{ (Check)}$$

$\eta_{Y|X}(0.5 \mid 0.8)$: Solve$_y$ for: $F_{Y|X}(y \mid 0.8) = 0.5$

$$\int_0^y \frac{30z^2+24}{34}\, dz = 0.5 \Rightarrow 10y^3 + 24y = 17 \Rightarrow y = 0.6126$$

$$\Rightarrow \eta_{Y|X}(0.5 \mid 0.8) = 0.6126.$$

More than 2 r.v.'s: $X_1, \dots, X_n$ $(n > 2)$

(Discrete) Joint pmf: $p(x_1, \dots, x_n) = P(X_1 = x_1, \dots, X_n = x_n)$ (≥ 0)

and $\sum_{x_1 \in D_{X_1}} \cdots \sum_{x_n \in D_{X_n}} p(x_1, \dots, x_n) = 1.$

(Conts.) Joint pdf: $f(x_1, \dots, x_n) \geq 0$

and $\int_{x_1=-\infty}^{\infty} \cdots \int_{x_n=-\infty}^{\infty} f(x_1, \dots, x_n)\, dx_n \dots dx_1 = 1$

and, $P(a_1 \le X_1 \le b_1, \dots, a_n \le X_n \le b_n) = \int_{x_1=a_1}^{b_1} \cdots \int_{x_n=a_n}^{b_n} f(x_1, \dots, x_n)\, dx_n \dots dx_1$

$X_1, X_2, \dots, X_n$ are indep. if:
For every subset $X_{i_1}, X_{i_2}, \dots, X_{i_k}$ of size k $(2 \le k \le n)$ of $X_1, \dots, X_n$:

$$p(x_{i_1}, \dots, x_{i_k}) = p_{X_{i_1}}(x_{i_1}) \cdots p_{X_{i_k}}(x_{i_k}) \quad \text{(Discrete)}$$

or, $f(x_{i_1}, \dots, x_{i_k}) = f_{X_{i_1}}(x_{i_1}) \cdots f_{X_{i_k}}(x_{i_k})$ (Conts.)

Sec 5.2: If $h(x,y)$ is a 2-dim fn.

$$E[h(X,Y)] = \begin{cases} \sum_{x \in D_X} \sum_{y \in D_Y} h(x,y)\, p(x,y). & \text{(Discrete)} \\ \int_{x \in D_X} \int_{y \in D_Y} h(x,y)\, f(x,y)\, dy\, dx. & \text{(Conts.)} \end{cases}$$

If $h(x,y) = h(x)$,

$$E[h(X,Y)] = \sum_{x \in D_X} \sum_{y \in D_Y} h(x) \cdot p(x,y)$$
$$= \sum_{x \in D_X} h(x) \sum_{y \in D_Y} p(x,y)$$
$$= \sum_{x \in D_X} h(x) \cdot p_X(x) = E[h(X)]$$

Eg 1: $h(x,y) = |x-y|$

$h(x,y) \mid$ ... 200

$E[h(X,Y)] = E|X-Y|$

$= 100 \times 0.2 + 0 \times 0.1 + 100 \times 0.2$

Eg 1: $h(x,y) = |x-y|$

$h(x,y)$ $x \backslash y$	0	100	200
100	100	0	100
250	250	150	50

$$E[h(X,Y)] = E|X-Y|$$
$$= 100 \times 0.2 + 0 \times 0.1 + 100 \times 0.2 + 250 \times 0.05 + 150 \times .15 + 50 \times .3$$
$$= 90 \quad \text{(Check)}$$

Dfn: $V[h(X,Y)] = E\Big[\underbrace{\{h(X,Y) - E[h(X,Y)]\}}_{K(X,Y)}{}^2\Big]$

$$= E[h^2(X,Y)] - E^2[h(X,Y)] \quad \text{(Check)}$$

Also, if $g(x,y) = h(x,y) + k(x,y)$,

$$E[g(X,Y)] = E[h(X,Y)] + E[k(X,Y)] \quad \text{(Check)}$$

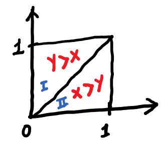
Eg 2: $h(x,y) = \frac{x}{2} + y + \frac{1}{2}$

$$E[h(X,Y)] = E\left[\frac{X}{2}\right] + E[Y] + \frac{1}{2}$$
$$= \frac{1}{2} \cdot E[X] + E[Y] + \frac{1}{2}$$
$$= \frac{1}{2} \cdot \frac{3}{5} + \frac{3}{5} + \frac{1}{2} \quad \text{(Check)}$$
$$= 1.4$$

Lecture 28

Monday, July 24, 2017 1:01 PM

Eg 2 :



$$E|X-Y| = \int_0^1\int_0^1 |x-y|\,\frac{6}{5}\cdot(x+y^2)\,dy\,dx$$

$$= \iint_{I} |x-y|\,\frac{6}{5}(x+y^2)\,dy\,dx + \iint_{II} |y-x|\,\frac{6}{5}(x+y^2)\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=x}^{1} (y-x)\,\frac{6}{5}\cdot(x+y^2)\,dy\,dx + \int_{x=0}^{1}\int_{y=0}^{x} (x-y)\,\frac{6}{5}\cdot(x+y^2)\,dy\,dx$$

(finish)

In general, to calculate $E|aX+bY+c|$ or $P(|aX+bY+c|>0)$, look at 2 sides of the straight line $aX+bY+c=0$.

• **Covariance:** (b/w 2 r.v.'s X and Y)

$$Cov(X,Y) \overset{def^n}{=} E\big[\underbrace{(X-\mu_X)(Y-\mu_Y)}_{h(x,y)}\big]$$

$$= \begin{cases} \sum\limits_{x\in D_X}\sum\limits_{y\in D_Y} (x-\mu_X)(y-\mu_Y)\,p(x,y) & \text{(Discrete)} \\ \int\limits_{x\in D_X}\int\limits_{y\in D_Y} (x-\mu_X)(y-\mu_Y)\,f(x,y)\,dy\,dx & \text{(Conts.)} \end{cases}$$

$$h(x,y) = (x-\mu_X)(y-\mu_Y) \begin{cases} >0, & \text{if } x>\mu_X \text{ and } y>\mu_Y \text{ or } x<\mu_X \text{ and } y<\mu_Y \\ <0, & \text{o.w.} \end{cases}$$

$$\Rightarrow Cov(X,Y) \begin{cases} >0 & \text{if } X\uparrow, Y\uparrow \text{ ; } X\downarrow, Y\downarrow \\ <0 & \text{if } X\uparrow, Y\downarrow \text{ ; } X\downarrow, Y\uparrow \end{cases}$$

Eg 1. $E(X) = 100\times .5 + 250\times 0.5 = 175$

$E(Y) = 125$ (Check)

$Cov(X,Y) = (100-175)(0-125)\times 0.2 + \ldots + (250-175)(200-125)\times 0.3$

$= 1875$ (Check)

Properties: 1. $Cov(X,Y) = Cov(Y,X)$

2. $Cov(X,Y) = E(XY) - E(X).E(Y).$

Pf: $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$= E[XY - \mu_X.Y - \mu_Y.X + \mu_X.\mu_Y]$

$= E(XY) - \mu_X.E(Y) - \mu_Y.E(X) + \mu_X.\mu_Y$

$= E(XY) - E(X).E(Y)$

3. $Cov(X,X) = E[(X-\mu_X)^2] = Var(X).$

3. $Cov(X,X) = E[(X-\mu_X)^2] = Var(X)$.

[Eg2: $E(X) = E(Y) = \frac{3}{5}$

$E(XY) = \int_0^1\int_0^1 xy \cdot \frac{6}{5}(x+y^2)\,dy\,dx = \frac{6}{5}\left(\int_0^1 x^2dx\right)\left(\int_0^1 y^2dy\right) + \frac{6}{5}\left(\int_0^1 x\,dx\right)\left(\int_0^1 y^3dy\right)$

$= \frac{7}{20}$ (Check)

$Cov(X,Y) = \frac{7}{20} - \frac{9}{25} = -0.01$]

4. $Cov(aX+b, Y) = a\,Cov(X,Y)$

Pf: $Cov(aX+b, Y) = E[(aX+\cancel{b} - a\mu_X - \cancel{b})(Y-\mu_Y)]$

$= a \cdot Cov(X,Y)$.

5. $Cov(aX+b, cY+d) = ac\,Cov(X,Y)$

Apply 4 for both variables.

If X & Y are measured in unit 1 and unit 2, then $Cov(X,Y)$ has unit unit 1 × unit 2.

Correlation:

$Corr(X,Y) \overset{Not^n}{=} \rho_{X,Y} \overset{Def^n}{=} \frac{Cov(X,Y)}{\sqrt{V(X) \cdot V(Y)}}$: Unit free [as $V(X)$ has unit $(\text{Unit } 1)^2$ and $V(Y)$ has unit $(\text{unit } 2)^2$.]

Properties: 1. $\rho_{X,Y} = \rho_{Y,X}$

2. $\rho_{X,X} = \frac{Cov(X,X)}{\sqrt{[V(X)]^2}} = \frac{V(X)}{V(X)} = 1$.

3. $\boxed{-1 \le \rho_{X,Y} \le 1}$

(i) $\rho_{X,Y} = 0$: Uncorrelated (NO linear association)

(ii) $|\rho_{X,Y}| = 1$: Perfect linear association

(iii) $|\rho_{X,Y}| \geq 0.8$: Strong association

(iv) $0.5 < |\rho_{X,Y}| < 0.8$: Moderate association

(v) $|\rho_{X,Y}| \le 0.5$: Weak association

$|\rho_{X,Y}|$: Strength of the association

Sign of $\rho_{X,Y}$: Direction

[Eg1. $Cov(X,Y) = 1875$

$V(X) = 5625$, $V(Y) = 6875$ (Check)

$\rho_{X,Y} = 0.301$ (Check)]

Lecture 29

Tuesday, July 25, 2017 1:00 PM

(4) $|\rho_{aX+b,cY+d}| = |\rho_{X,Y}|$

Pf. $Cov(aX+b, cY+d) = ac \cdot Cov(X,Y)$

$V(aX+b) = a^2 V(X)$

$V(cY+d) = c^2 V(Y)$

$$\rho_{aX+b,cY+d} = \frac{Cov(aX+b,cY+d)}{\sqrt{V(aX+b) \cdot V(cY+d)}}$$

$$= \frac{ac \cdot Cov(X,Y)}{\sqrt{a^2 V(X) \cdot c^2 V(Y)}} = \frac{ac}{|a||c|} \cdot \frac{Cov(X,Y)}{\sqrt{V(X) \cdot V(Y)}}$$

$$= \frac{ac}{|a| \cdot |c|} \cdot \rho_{X,Y}$$

$|\rho_{aX+b,cY+d}| = |\rho_{X,Y}|$

Moreover, $\rho_{aX+b,cY+d} = \pm \rho_{X,Y}$ ['+' if a and c have the same sign, '-' if a and c have opposite signs]

(5) $\rho_{aX+b,X} = \pm \rho_{X,X} = \pm 1$ ('+' if $a>0$, '-' if $a<0$)

Use 4 with $Y=X$ and $c=1$ and $d=0$.

Sp. case: $\rho_{X,-X} = -1 = \rho_{-2X+3,X}$

Q: What if $a=0$? (If $a=0$, $aX+b=b$ w.p. 1)

$$\rho_{aX+b,X} = \rho_{b,X} = \frac{Cov(X,b)}{\sqrt{\sigma_X^2 \cdot \sigma_b^2}}$$

$$= \frac{0}{\sigma_X \cdot 0}$$ $\left[\begin{array}{l} \sigma_b^2 = 0 \\ Cov(X,b) = E[(X-\mu_X)(b-b)] = 0 \end{array} \right]$

: NOT defined

Fact: $\rho_{X,Y}$ is NOT defined if X or Y is degenerate.

Eg 2: $Cov(X,Y) = -0.01$

$E(X) = E(Y) = \frac{3}{5}$

$V(X) = \frac{11}{150}$, $V(Y) = \frac{2}{250}$ (Check)

$$\rho_{X,Y} = -\frac{0.01}{\sqrt{\frac{11}{150} \times \frac{2}{250}}} = -0.131$$

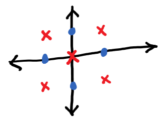
- Comparison of Indep. & Uncorrelatedness:

a. Indep. $\Rightarrow$ Uncorrelated

b. Uncorrelated $\not\Rightarrow$ Indep.

a. Indep. $\Rightarrow E(XY) = E(X).E(Y) \Rightarrow Cov(X,Y) = 0 \Rightarrow \rho_{X,Y} = 0$.

b. Counter Example:



$D = \{(-1,0),(0,1),(0,-1),(1,0)\}$

$$p(x,y) = \begin{cases} 0.25, & \text{if } (x,y) \in D \\ 0, & \text{o.w.} \end{cases}$$

$D_X = \{-1,0,1\} = D_Y$

Non-rectangular support $\Rightarrow$ NOT indep. (Look at any $\underbrace{\times}_{}$)

$p(x,y) = 0 \neq \underset{0}{\underset{\vee}{p_X(x)}} . \underset{0}{\underset{\vee}{p_Y(y)}}$

e.g. $p(0,0) = 0 \neq p_X(0).\,p_Y(0) = 0.5 \times 0.5$

$XY = 0$ w.p. 1 $\Rightarrow E(XY) = 0$

Also, $p_X(x) = \begin{cases} .25, & \text{if } x=-1 \\ .5, & \text{if } x=0 \\ .25, & \text{if } x=1 \end{cases} \Rightarrow E(X) = 0$

Similarly, $E(Y) = 0$.

$\Rightarrow Cov(X,Y) = 0 - 0.0 = 0 \Rightarrow \rho_{X,Y} = 0$. (clearly, $\sigma_X, \sigma_Y > 0$)

$\boxed{\text{Correlation} \not\Rightarrow \text{Causation}}$

- Uniform Dist^n for (X,Y): (conts.)

$f(x,y) = K\, I_A(x,y)$, $D = A$

$$\iint_{\mathbb{R}^2} f(x,y)\,dy\,dx = \iint_A K.\,dy\,dx = K \iint_A 1.\,dy\,dx = K|A| = 1$$

$$\Rightarrow K = \frac{1}{|A|}$$

$$P(B) = \begin{cases} \dfrac{|B|}{|A|}, & \text{if } B \subset A \\ \dfrac{|B \cap A|}{|A|}, & \text{in general} \end{cases} \qquad [\,|A| = area(A)\,]$$

- Sec 5.5: A linear combination of $X_1, \ldots, X_n$ (RV's) with means $\mu_1, \mu_2, \ldots, \mu_n$ and SD's $\sigma_1, \sigma_2, \ldots, \sigma_n$ is defined as:

$$a_1X_1 + a_2X_2 + \ldots + a_nX_n \text{ for constants } a_1, \ldots, a_n.$$

1. $E(a_1X_1 + \ldots + a_nX_n) = a_1E(X_1) + \ldots + a_nE(X_n) = a_1\mu_1 + \ldots + a_n\mu_n$
2. $V(a_1X_1 + \ldots + a_nX_n) = a_1^2V(X_1) + \ldots + a_n^2V(X_n) = a_1^2\sigma_1^2 + \ldots + a_n^2\sigma_n^2$ [If $X_1, \ldots, X_n$ are indep.]
3. $V(a_1X_1 + \ldots + a_nX_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_ia_j\, Cov(X_i, X_j)$
4. If $X_i \sim N(\mu_i, \sigma_i^2)$ are indep,

$$a_1X_1 + \ldots + a_nX_n \sim N(a_1\mu_1 + \ldots + a_n\mu_n,\ a_1^2\sigma_1^2 + \ldots + a_n^2\sigma_n^2).$$

e.g. $E(X_1 - X_2) = E(X_1) - E(X_2)$ $\Big\}$ Take $n=2, a_1=1, a_2=-1$ in 1 and 2

$V(X_1 - X_2) = V(X_1) + V(X_2)$

$a_1X_1 + \dots + a_nX_n \sim N($

e.g. $E(X_1 - X_2) = E(X_1) - E(X_2)$

$V(X_1 - X_2) = V(X_1) + V(X_2)$ (if X_1 and X_2 are indep.)

} Take $n=2, a_1 = 1, a_2 = -1$ in 1 and 2

Moreover if $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ are indep.,

$X_1 - X_2 \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$ (By 4)

Sketches of Proofs:

1. For $n=2$,

$$\begin{aligned} E(a_1X_1 + a_2X_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (a_1x_1 + a_2x_2) f(x_1,x_2)\, dx_2\, dx_1 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_1x_1 f(x_1,x_2)\, dx_2\, dx_1 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_2x_2 f(x_1,x_2)\, dx_2\, dx_1 \\ &= a_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 f(x_1,x_2)\, dx_2\, dx_1 + a_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 \cdot f(x_1,x_2)\, dx_2\, dx_1 \\ &= a_1 E(X_1) + a_2 E(X_2) \end{aligned}$$

} Similar proof in the discrete case.

For $n=3$, $E(a_1X_1 + a_2X_2 + a_3X_3) = E(Y + a_3X_3) = E(Y) + E(a_3X_3)$ [By using result for $n=2$ with $a_1 = 1, X_1 = Y, a_2 = a_3, X_2 = X_3$]

(Let $Y = a_1X_1 + a_2X_2$) $= E(a_1X_1 + a_2X_2) + a_3E(X_3)$

$= a_1E(X_1) + a_2E(X_2) + a_3E(X_3)$ [Using result for $n=2$ again]

Proof by induction.

3.
$$\begin{aligned} V(aX + bY) &= E[(aX+bY)^2] - [E(aX+bY)]^2 \\ &= E[a^2X^2 + b^2Y^2 + 2abXY] - [aE(X) + bE(Y)]^2 \\ &= a^2E(X^2) + b^2E(Y^2) + 2abE(XY) - a^2E^2(X) - b^2E^2(Y) - 2abE(X).E(Y) \\ &= a^2\{E(X^2) - E^2(X)\} + b^2\{E(Y^2) - E^2(Y)\} + 2ab\{E(XY) - E(X).E(Y)\} \\ &= a^2V(X) + b^2V(Y) + \underbrace{2ab\,Cov(X,Y)}_{ab\,cov(X,Y) + ba\,cov(Y,X)} \end{aligned}$$

Similar proof by induction.

2. If $X_1, \dots, X_n$ are indep., $Cov(X_i, X_j) = 0 \quad \forall i \neq j$

By 3, $V(a_1X_1 + a_2X_2 + \dots + a_nX_n) = \sum_{i=j}\sum a_ia_j Cov(X_i, X_j) = \sum_{i=1}^{n} a_i^2 Cov(X_i, X_i) = \sum_{i=1}^{n} a_i^2 V(X_i)$

Proof of 4 is beyond the scope of this book.